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ASYMPTOTIC DISTRIBUTIONS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH WIDE BAND NOISE INPUTS; APPROXIMATE INVARIANT MEASURES.[†]

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January 14, 1981

ABSTRACT

Let $\{x^\varepsilon(\cdot)\}$ be a sequence of solutions to an ordinary differential equation with random right sides (due to input noise $\{\xi^\varepsilon(\cdot)\}$) and which converges weakly to a diffusion $x(\cdot)$ with unique invariant measure $\mu(\cdot)$. Let $\mu(t,\cdot)$ denote the measure of $x(t)$, and suppose that $\mu(t,\cdot) \to \mu(\cdot)$ weakly. The paper shows, under reasonable conditions, that the measures of $x^\varepsilon(t)$ are close to $\mu(\cdot)$ for large t and small ε. In applications, such information is often more useful than the simple fact of the weak convergence. The noise $\xi^\varepsilon(\cdot)$ need not be bounded, the pair $(x^\varepsilon(\cdot), \xi(\cdot))$ need not be Markovian (except for the unbounded noise case), and the dynamical terms need not be smooth. The discrete parameter case is treated, and several examples arising in control and communication theory are given.

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I. INTRODUCTION

Many results are available concerning the weak convergence of a sequence of processes $\{x^\epsilon(\cdot)\}$ to a diffusion $x(\cdot)$ (with values in R^r), where $\{x^\epsilon(\cdot)\}$ are the solutions to ordinary differential equations with random right hand sides [1]-[5]; e.g., where $\dot{x}^\epsilon = F_\epsilon(x^\epsilon, \xi^\epsilon)$ for some function $F_\epsilon(\cdot, \cdot)$ and a "wide band" noise process $\xi^\epsilon(\cdot)$. For small $\epsilon > 0$, the weak convergence basically gives us information on the approximation of $x^\epsilon(\cdot)$ by $x(\cdot)$ on arbitrarily large but still finite time intervals. In applications to control and communication theory, information concerning the <u>asymptotic behavior</u> - for small ϵ, but for "large" t is often of much greater interest. For example, we want to know how well the asymptotic distributions (the distributions at arbitrarily large times) of the $x^\epsilon(\cdot)$ are approximated by the (say) invariant measure of $x(\cdot)$ for small ϵ.

This problem was discussed in [2, Section 6] and, roughly speaking, a result of the following type was obtained. Let $\xi(\cdot)$ be a stationary finite state jump Markov process, define $\xi^\epsilon(t) = \xi(t/\epsilon^2)$ and for smooth functions $F(\cdot,\cdot)$, $G(\cdot,\cdot)$, $\bar{G}(\cdot)$, where $EF(x,\xi(t)) \equiv 0 \equiv EG(x,\xi(t))$, define $x^\epsilon(\cdot)$ by

$$(1.1) \qquad \dot{x}^\epsilon = F(x,\xi^\epsilon)/\epsilon + G(x,\xi^\epsilon) + \bar{G}(x), \quad x^\epsilon(t) \in R^r \quad .$$

Then, if $x^\epsilon(0)$ converges weakly to $x(0)$, $\{x^\epsilon(\cdot)\}$ converges weakly to a diffusion $x(\cdot)$ whose generator $\mathscr{L}$ is defined by

$$(1.2) \qquad \mathscr{L}f(x) = f'_x(x)\bar{G}(x) + \int_0^\infty E(f'_x(x)F(x,\xi(t)))'_x F(x,\xi(0))dt \quad .$$

Suppose that $x(\cdot)$ has a unique invariant measure $\mu(\cdot)$, and let there be a smooth Liapunov function $0 \leq V(x) \to \infty$ as $|x| \to \infty$, and a $\gamma > 0$ such that

(1.3) $\mathscr{L}V(x) \leq -\gamma V(x)$,   for large $|x|$.

Then, for small ε, $(x^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot))$ has an invariant measure $\nu^{\varepsilon}(\cdot)$ whose x-marginals $\mu^{\varepsilon}(\cdot)$ converge weakly to $\mu(\cdot)$ as $\varepsilon \to 0$ [2, Section 6].

We are interested in results of the same type, but where $\xi^{\varepsilon}(\cdot)$ might not be Markov - or even bounded (Gaussian, for example), and where $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ might not be smooth. Also, in many interesting applications (1.3) does not hold (e.g., in many cases where $F(\cdot,\cdot)$, $G(\cdot,\cdot)$ and $\bar{G}(\cdot)$ are bounded). Thus, for our problem $\{x^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot)\}$ might not have an invariant measure, even for small $\varepsilon > 0$. Furthermore, we are also interested in the case where the evolution of $\xi^{\varepsilon}(\cdot)$ depends on $x^{\varepsilon}(\cdot)$. Also, [2] does not address the question of the distributions of $x^{\varepsilon}(t)$ being close to $\mu(\cdot)$ for large t and small ε and arbitrary initial condition, a problem which is important in applications. The basic techniques used here are similar to those in [2]; both heavily depending on the use of "averaged Liapunov functions."

Section 2 contains the basic approximation theorem, but using the condition (A5) which is not usually directly verifiable. A verifiable condition for (A5) appears in Section 4.
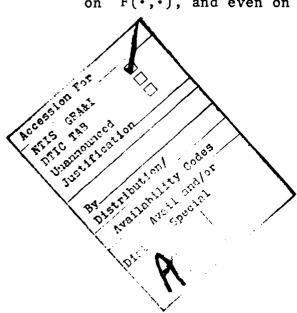
Since our interest is only in the asymptotic properties of the

measures of the $\{x^\epsilon(\cdot)\}$ for small ϵ, we assume weak convergence (A4) - preferring to keep close to the main purpose. In any case, many known sets of conditions imply the weak convergence (see [1]-[5] or the conditions in Section 4). Section 4 also discusses the changes required when $F(\cdot,\cdot)$ or $G(\cdot,\cdot)$ are not smooth, a case which is important in many applications. The "unbounded" noise case is dealt with in Section 5, some examples are discussed in Section 6, and in Section 7 we treat the discrete parameter case, when the noise might be "state" dependent. Section 3 contains a result that is useful in the proof of Section 4.

For future reference, we note that if $\xi(\cdot)$ is Markov, it is sometimes convenient to write (1.2) in the form

$$(1.4) \qquad \mathscr{L}f(x) = f'_x(x)\bar{G}(x) + \int_0^\infty E(f'_x(x)E_{\xi(0)}F(x,\xi(t)))'_x F(x,\xi(0))ds .$$

The advantage of (1.4) is that if $F(\cdot,\cdot)$ is not differentiable, then the derivative in (1.2) might not exist, but that in (1.4) might exist for $t > 0$, and we can weaken the conditions required on $F(\cdot,\cdot)$, and even on $G(\cdot,\cdot)$.

2. THE BASIC CONVERGENCE THEOREM

Assumptions

A1. $x(\cdot)$ is a Feller diffusion process with continuous coefficients and a unique weak sense solution on $[0,\infty)$ for each intial condition.

A2. $x(\cdot)$ has a unique invariant measure $\mu(\cdot)$ and $\mu(x,t,\cdot)$, the measure of $x(t)$ when $x(0) = x$, converges weakly to $\mu(\cdot)$ for each x, as $t \to \infty$.

A3.   The convergence in (A2) is uniform in compact x-sets; i.e., for each $f(\cdot) \in \mathscr{C}_r$, the space of continuous bounded functions on R^r, $E_x f(x(t)) \to E_\mu f(x(0))$ uniformly in x in compact sets, where E_μ denotes expectation with respect to the stationary measure of $x(\cdot)$. ((A3) need only hold for the functions of interest in the Theorem.)

A4. $x^\epsilon(\cdot) \to x(\cdot)$ weakly (initial condition $x(0)$) if $x^\epsilon(0) \to x(0)$ weakly in $D^r[0,\infty)$.

A5.   There is an $\epsilon_0 > 0$ such that $\{x^\epsilon(t), 0 < \epsilon \leq \epsilon_0, t \geq 0\}$ is tight. (See Theorem 3 for a verifiable criterion for (A5).)

Remark on (A3).   Condition (A3) is implied if $\bar{\mu}$ is weakly stable in the sense of Liapunov. More specifically, let $\{f_n\}$ be a sequence of continuous and uniformly bounded functions, each with compact support such that $\mu_\alpha \to \mu_0$ weakly (as $\alpha \to 0$) is

equivalent to $\int f_n(y)\mu_\alpha(dy) \to \int f_n(y)\mu_0(dy)$ for each n. Let $\mu(t,\cdot)$ denote the distribution of $x(t)$. Suppose that for each $\delta > 0$ and $f, \ldots, f_m$, there are $\epsilon > 0$ and $n < \infty$ such that

$$\left| \int f_i(y)\mu(0,dy) - \int f_i(y)\mu(dy) \right| < \epsilon, \quad i \leq n$$

implies

$$\left| \int f_i(y)\mu(t,dy) - \int f_i(y)\mu(dy) \right| < \delta, \quad i \leq m, \quad \text{all} \quad t \geq 0 \quad .$$

Then (A4) holds. In the completely degenerate case, where $x(\cdot)$ satisfies an ordinary differential equation and μ is concentrated at a point x_0, then the above criterion is equivalent to the usual Liapunov stability of x_0.

Lemma 1. For any integer m, let $f(\cdot) \in \mathscr{L}_{mr+r}$. Assume (A1)-(A3) and let $0 = \Delta_0 < \Delta_1 \ldots < \Delta_m$. Let S denote a tight set of R^r valued random variables. Then

$$E_{x(0)} f(x(t + \Delta_i), \ i \leq m) \to E_\mu f(x(\Delta_i), \ i \leq m)$$

uniformly for $x(0) \in S$, as $t \to \infty$.

The result is a consequence of (A1)-(A3), and the proof is omitted. The uniformity, in particular, requires (A3). The Δ_i in Theorem 1 are the same as those in Lemma 1.

Theorem 1. Assume (A1)-(A5). Then for each $f(\cdot) \in \mathscr{L}_{rm+r}$ and

$\delta > 0$, <u>there are</u> $t_0(f,\delta) < \infty$ <u>and</u> $\varepsilon_0(f,\delta) > 0$ <u>such that for</u> <u>all</u> $t \geq t_0(f,\delta)$ <u>and</u> $\varepsilon \leq \varepsilon_0(f,\delta)$, <u>and any sequence</u> $\{x^\varepsilon(\cdot)\}$ <u>which converges weakly to</u> $x(\cdot)$,

(2.1) $|Ef(x^\varepsilon(t + \Delta_i), i \leq m) - E_\mu f(x(\Delta_i), i \leq m)| < \delta$.

<u>Let</u> $\xi(\cdot)$ <u>be Markov and</u> $(x^\varepsilon(\cdot), \xi^\varepsilon(\cdot))$ <u>Markov and have an invari-</u> <u>ant measure</u> $v^\varepsilon(\cdot)$. <u>Replace</u> (A5) <u>by:</u> <u>There is a sequence</u> $T_\varepsilon \to \infty$ (T_ε <u>can depend on the initial condition</u>) <u>such that</u> (A5) <u>holds</u> <u>for</u> $t \geq T_\varepsilon$. <u>Then</u> $\{v^\varepsilon(\cdot),$ <u>small</u> $\varepsilon\}$ <u>is tight and its x-marginals</u> $\{\mu^\varepsilon(\cdot)\}$ <u>converge weakly to</u> $\mu(\cdot)$.

<u>Remark</u>. The theorem implies that the convergence as $t \to \infty$ in (2.1) is uniform in ε for small ε, a fact which is important in applications. In applications, it is often possible to prove results such as $\lim_{t\to\infty} E_x|x^\varepsilon(t)| \leq K$, where K does not depend on ε or x. Then if $\{\xi^\varepsilon(\cdot)\}$ is bounded, the replacement for (A5) in the last paragraph holds.

<u>Proof</u>. Suppose that (2.1) is false. Then there is a subsequence $\varepsilon \to 0$ and a sequence $\{t_\varepsilon\} \to \infty$ such that

(2.2) $|Ef(x^\varepsilon(t_\varepsilon + \Delta_i), i \leq m) - E_\mu f(x(\Delta_i), i \leq m)| \geq \delta > 0$.

We will find a further subsequence, also indexed by ϵ, which violates (2.2). Fix $T > 0$. By (A5), we can choose a further subsequence such that $\{x^\epsilon(t_\epsilon - T)\}$ converges weakly to a random variable $x(0)$. By (A4), $\{x^\epsilon(t_\epsilon - T + \cdot)\}$ converges weakly to $x(\cdot)$ with initial condition $x(0)$ and

$$(2.3) \qquad Ef(x^\epsilon(t_\epsilon - T + T + \Delta_i), \ i \le m) \to EE_{x(0)} f(x(T + \Delta_i), \ i \le m).$$

By (A5), the set S of all possible $x(0)$ (over all $T > 0$ and weakly convergent subsequences) is tight. By Lemma 1, we can take T large enough such that

$$(2.4) \qquad |EE_{x(0)} f(x(T + \Delta_i), \ i \le m) - E_\mu f(x(\Delta_i), \ i \le m)| < \delta/2.$$

Equations (2.3) and (2.4) contradict (2.2).

The proof of the last assertion is similar to the last part of the proof of Theorem 4 and is omitted.

Q.E.D.

3. A LIAPUNOV FUNCTION CRITERION FOR (3.1)

(3.1) $\bigcup_{x \in B} \{x(t), \; t \geq 0 \; , \; x(0) = x\}$ is tight for each compact B.

To prove (3.1), we will require condition (A6).

A6. There is a continuous Liapunov function $0 \leq V(x) \to \infty$ as $|x| \to \infty$ and a λ_0 and $\alpha_0 > 0$ such that $V(x) \leq - \alpha_0$ for $x \notin Q_0 = \{x : V(x) \leq \lambda_0\}$. The partial derivatives of $V(\cdot)$ up to order 2 are continuous.

Theorem 2 is proved partially because the proof is a prototype of the technique used later to verify (A5).

Theorem 2. Under (A1) and (A6), condition (3.1) holds.

Proof. (A6) implies that Q_0 is a recurrence set for $x(\cdot)$ [6]. We suppose w.l.o.g. that $\min_x V(x) = 0$.

Let $\lambda_2 > \lambda_1 > \lambda_0$ and define $Q_i = \{x : V(x) \leq \lambda_i\}$. Let τ_0 denote a Markov time such that $x(\tau_0) \notin Q_0$, and define $\tau_1 = \min\{t : t > \tau_0, \; x(t) \in Q_0\}$. Then ([6]) an application of Ito's formula yields that for any Markov time $t \geq \tau_0$ for which $E_{x(\tau_0)}(t - \tau_0) < \infty$:

$$(3.2) \quad E_{x(\tau_0)} V(x(t \cap \tau_1)) - V(x(\tau_0)) \leq E_{x(\tau_0)} \int_{\tau_0}^{t \cap \tau_1} \mathscr{L}V(x(s))ds$$

$$\leq - \alpha_0 E_{x(\tau_0)}(t \cap \tau_1 - \tau_0) \; ,$$

$$(3.3) \quad E_{x(\tau_0)}(\tau_1 - \tau_0) \leq V(x(\tau_0))/\alpha_0 \; , \quad P_{x(\tau_0)}\{\tau_1 < \infty\} = 1 \; ,$$

(3.4) $\quad P_{x(\tau_0)}\{\sup_{\tau_1 \geq s \geq \tau_0} V(x(s)) \geq \lambda\} \leq V(x(\tau_0))/\lambda$.

Define $\quad T_0 = \lambda_1/\alpha_0$.

To get (3.1), define a sequence of Markov times $\{\sigma_n\}$ as follows. For n = 1,

$$\sigma_1 = \min\{t : x(t) \in Q_0\} .$$

For n > 1,

$$\sigma_n = \sigma_{n-1} + T_0 \quad \text{if} \quad x(t) \in Q_1^O \equiv Q_1 - \partial Q_1 \quad \text{for} \quad t \in [\sigma_{n-1}, \sigma_{n-1} + T_0],$$

$$\sigma_n = \inf\{t : t > \sigma_{n-1}, x(t) \in \partial Q_1\} \quad \text{if} \quad x(\sigma_n) \in Q_1^O \quad \text{but}$$

$$x(t) \in \partial Q_1 \quad \text{for some} \quad t \in [\sigma_{n-1}, \sigma_{n-1} + T_0] .$$

$$\sigma_n = \inf\{t : t > \sigma_{n-1}, x(t) \in Q_0\} \quad \text{if} \quad x(\sigma_{n-1}) \notin Q_1^O$$

For $n \geq 1$, $E_{x(\sigma_n)}(\sigma_{n+1} - \sigma_n) \leq T_0$ and $P_x\{\sigma_1 < \infty\} = .1$. In fact if the set B in (3.1) is contained in Q_2, then $\sup_{x \in B} E_x \sigma_1 \leq \lambda_2/\alpha_0$ by (3.3). Fix $\delta > 0$. For each k and t define j(t,k) by

$$\sigma_{j(t,k)} = \min\{\sigma_i : P_{x(\sigma_i)}\{t \geq \sigma_{k+i}\} \leq \delta/4\} .$$

Define the intervals $A_i = [\sigma_{j(t,k)+(i-1)}, \sigma_{j(t,k)+i}]$, i = 1, ...,k. Then for $\lambda \geq \lambda_1$,

(3.5) $\quad P_x\{V(x(t)) \geq \lambda\} \leq P_x\{\sigma_{j(t,k)} \geq t\} + \sum_{j=1}^{k} P_x\{\sup_{s \in A_i} V(x(s)) \geq \lambda\}$

$$+ \delta/4 .$$

We can choose k such that the first term on the right hand side
is $\leq \delta/4$ for all $x \in B$. Then, by (3.4), we can choose λ such
that the sum is $\leq \delta/4$ for all x. Since $V(x) \to \infty$ as $|x| \to \infty$,
(3.1) holds.   Q.E.D.

4. An Averaged Liapunov Function Criterion for (A5).

In this section, we use the model (1.1) and a strong mixing condition on $\xi(\cdot)$. The development should be viewed as an illustration of a general technique. The mixing condition is too strong for many applications, and other conditions are considered in Section 5. The mixing condition is used simply to assure certain bounds. In specific cases, a very similar development can be carried through under other conditions on the noise, and the same bounds shown. Also, (see, e.g., Example 2 in Section 6), a very similar development can often be carried through for equations of forms other than (1.1). The smoothness requirement in Condition (B2) is weakened in the remarks after the proof. In order to get the necessary inequalities for any Liapunov function based approach, an assumption such as (B4) seems to be required. The conditions hold in numerous cases of interest.

We will use

B1. $\xi(\cdot)$ is a bounded, right continuous, stationary ϕ-mixing process [7] with $\int_0^\infty \phi^{1/2}(t)dt < \infty$.

B2. $F(\cdot,\cdot)$, $G(\cdot,\cdot)$ and $\bar{G}(\cdot)$ are continuous, R^r valued functions whose growth (as $|x| \to \infty$) is $O(|x|)$. The partial derivatives of $F(\cdot,\xi)$ up to order 2 (and of $G(\cdot,\xi)$ up to order 1) are bounded uniformly in x,ξ, and $EF(x,\xi) \equiv 0 \equiv EG(x,\xi)$.

B3. <u>There is a diffusion process</u> $x(\cdot)$ <u>with differential genera-</u> <u>tor</u> $\mathscr{L}$ <u>defined by</u> (1.2), <u>and which satisfies</u> (A1)-(A3). <u>Also,</u> (A6) <u>holds, but the partial derivatives of</u> $V(\cdot)$ <u>up</u> <u>to order</u> 3 <u>are continuous.</u>

B4. <u>There are constants</u> K <u>such that, uniformly in</u> x, ξ,

(4.1a) $\qquad |V'_x(x)G(x,\xi)| + |V'_x(x)F(x,\xi)| \le (1 + V(x))$

(4.1b) $\qquad |(V'_x(x)F(x,\xi))'_x F(x,\xi)| \le K(1 + V(x))$

(4.2) $\qquad |(V'_x(x)G(x,\xi))'_x U(x,\xi)| \le K(1 + |\mathscr{L}V(x)|)$
$$\text{for} \quad U = F, G, \bar{G} \quad .$$

(4.3) $\qquad |(V'_x(x)F(x,\xi))'_x U(x,\xi)| \le K(1 + |\mathscr{L}V(x)|), \quad U = G, \bar{G}$

(4.4) $\qquad |((V'_x(x)F(x,\xi))'_x F(x,\xi))'_x U(x,\xi)| \le K(1 + |\mathscr{L}V(x)|) \quad ,$
$$U = F, G, \bar{G} \quad .$$

Note that here and in the sequel the value of K might change from usage to usage.

Define the differential operator $\hat{A}^\varepsilon$ and its domain $\mathscr{D}(\hat{A}^\varepsilon)$ as in [4],[5], [8]. The method of use of $\hat{A}^\varepsilon$ is similar to that in [4] and the averaging method is similar to that in [2], [4], [5].

<u>Theorem 3.</u> <u>Under</u> (B1)-(B4) <u>and the tightness of</u> $\{x^\varepsilon(0)\}$, <u>condi-</u> (A5) <u>holds.</u>

Proof. For each integer N, define $S_N = \{x: |x| \leq N\}$. Let $b_N(\cdot)$ denote a function with values in $[0,1]$ and satisfying $b_N(x) = 1$ in S_N and equals 0 in $R^r - S_{N+1}$, and the partial derivatives up to order 3 are bounded uniformly in N,x. Define $V_N(\cdot) = V(\cdot)b_N(\cdot)$. The $V(\cdot)$ is "truncated" because we cannot apply $\hat{A}^\epsilon$ to unbounded functions without additional conditions. $V(\cdot)$ is the Liapunov function which is to be "averaged," and $\hat{A}^\epsilon$ plays the role of a "differential" generator for $x(\cdot)$. Let E_t^ϵ denote expectation conditioned on $\xi^\epsilon(s)$, $s \leq t$, $x^\epsilon(0)$. We have (writing $x = x^\epsilon(\cdot)$)

$$(4.5) \qquad \hat{A}^\epsilon V_N(x) = V'_{N,x}(x) \left[\frac{F(x,\xi^\epsilon(t))}{\epsilon} + G(x,\xi^\epsilon(t)) + \bar{G}(x) \right] .$$

To average out $V'_{N,x}(x) G(x,\xi^\epsilon(t))$, define $V_{N,0}^\epsilon(t) = V_{N,0}^\epsilon(x^\epsilon(t),t)$, where

$$V_{N,0}^\epsilon(x,t) = \int_0^\infty V'_{N,x} E_t^\epsilon G(x,\xi^\epsilon(t+s))ds .$$

By changing variables $s/\epsilon^2 \to s$ and using (B1) and (4.1), we get (the K do not depend on N)

$$(4.6) \qquad |V_{N,0}^\epsilon(x,t)| \leq \epsilon K(1 + V(x)), \quad x \in S_N .$$

We have $V_{N,0}^\epsilon(\cdot) \in \mathscr{D}(\hat{A}^\epsilon)$ and (write $x = x^\epsilon(t)$)

$$A^\epsilon V_{N,0}^\epsilon(x,t) = - V'_{N,x}(x) G(x,\xi^\epsilon(t))$$

$$+ \int_0^\infty ds \left\{ E_t^\epsilon V'_{N,x}(x) G(x,\xi^\epsilon(t+s)) \right\}'_x \left\{ \bar{G}(x) + G(x,\xi^\epsilon(t)) + \frac{F(x,\xi^\epsilon(t))}{\epsilon} \right\}$$

By changing variables $s/\varepsilon^2 \to s$ and using (B1) and (4.2), we get that the integral is bounded by

(4.8) $\qquad \varepsilon\, K(1 + |\mathscr{Y}V(x)|), \quad x \in S_N$.

We now proceed to average out the $V'_{N,x}(x)\, F(x,\xi^\varepsilon(t))/\varepsilon$ component of (4.5). Define $V^\varepsilon_{N,1}(t) = V^\varepsilon_{N,1}(x^\varepsilon(t),t)$, where

$$V^\varepsilon_{N,1}(x,t) = \int_0^\infty V'_{N,x}(x)\, E^\varepsilon_t\, F(x,\xi^\varepsilon(t+s))ds/\varepsilon$$

By (B1) and (4.1), and the change of variables $s/\varepsilon^2 \to s$, $V^\varepsilon_{N,1}(x,t)$ satisfies the bound in (4.6). Also, $V_{N,1}(\cdot) \in \mathscr{D}(\hat{A}^\varepsilon)$ and (write $x = x^\varepsilon(t)$)

(4.9) $\qquad \hat{A}^\varepsilon V^\varepsilon_{N,1}(x,t) = - V'_{N,x}(x)\, F(x,\xi^\varepsilon(t))/\varepsilon\ +$

$$\int_0^\infty ds \left\{ V'_{N,x}(x)\, E^\varepsilon_t\, \frac{F(x,\xi^\varepsilon(t+s))}{\varepsilon} \right\}'_x \left\{ \frac{F(x,\xi^\varepsilon(t))}{\varepsilon} + G(x,\xi^\varepsilon(t)) \right.$$

$$\left. + \bar{G}(x) \right\} \ .$$

By (B1), (4.3) and the change of variables $s/\varepsilon^2 \to s$, the terms in (4.9) involving $G + \bar{G}$ are bounded by (4.8).

The remaining term in (4.9) must now be averaged out. Define $V^\varepsilon_{N,2}(t) = V^\varepsilon_{N,2}(x^\varepsilon(t),t)$ by

$$(4.10) \quad V_{N,2}^{\varepsilon}(x,t) = \int_0^{\infty} d\tau \int_0^{\infty} ds \left\{ (V_{N,x}'(x) \ E_t^{\varepsilon} \ F(x,\xi^{\varepsilon}(t+s+\tau)))_x' F(x,\xi^{\varepsilon}(t+\tau)) \right.$$

$$\left. - E(V_{N,x}'(x) \ F(x,\xi^{\varepsilon}(t+s+\tau)))_x' F(x,\xi^{\varepsilon}(t+\tau)) \right\}/\varepsilon^2 \quad .$$

By (B1), (4.3) and the change of variables $s/\varepsilon^2 \to s$, $\tau/\varepsilon^2 \to \tau$, $V_{N,2}^{\varepsilon}(x,t)$ is bounded by (4.6). Also, $V_{N,2}^{\varepsilon}(\cdot) \in \mathscr{D}(\hat{A}^{\varepsilon})$ and (writing $x = x^{\varepsilon}(t)$)

$$(4.11) \quad \hat{A}^{\varepsilon} V_{N,2}^{\varepsilon}(x,t) = - \text{ (inner integral of (4.10) evaluated at}$$

$$\tau = 0) + (V_{N,2}^{\varepsilon}(x,t))_x' \ \dot{x}^{\varepsilon} \quad .$$

By (B1), (4.4), and the usual change of variables, the last term on the right side of (4.11) is bounded by (4.8).

Define $V_N^{\varepsilon}(x,t) = V_N(x) + \sum_{i=0}^{2} V_{N,i}^{\varepsilon}(x,t)$ and $V_N^{\varepsilon}(t) = V_N^{\varepsilon}(x^{\varepsilon}(t),t)$. Define $V^{\varepsilon}(x,t) = V_{\infty}^{\varepsilon}(x,t)$. For $x \in S_N$,

$$(4.12a) \quad |V_N^{\varepsilon}(x,t) - V_N(x)| \leq \varepsilon \ K(1 + V(x))$$

$$(4.12b) \quad \hat{A}^{\varepsilon} V^{\varepsilon}(t) = \mathscr{L} V_N(x^{\varepsilon}(t)) + O(\varepsilon)(1 + |\mathscr{L} V(x^{\varepsilon}(t))|) \quad .$$

Also, $V^{\varepsilon}(x,t) \geq - K\varepsilon$, and the expression in (4.12b) is $\leq - \alpha_0/2$ in $S_N - Q_0$ for small $\varepsilon > 0$.

Let τ_0 denote a stopping time such that $x^{\varepsilon}(\tau_0) \notin Q_0$ and define

$$\tau_1 = \inf\{t: t \geq \tau_0, \ x^{\varepsilon}(t) \in Q_0\} \ ,$$

$$\tau_N = \inf\{t: x^{\varepsilon}(t) \in S_N\} \quad .$$

Then, for small enough $\varepsilon > 0$, and τ any stopping time satisfying $\tau \geq \tau_0$ and $E(\tau - \tau_0) < \infty$,

$$(4.13) \quad E^{\varepsilon}_{\tau_0 \cap \tau_N} V^{\varepsilon}_N(x^{\varepsilon}(\tau_1 \cap \tau_N \cap \tau), \; \tau_1 \cap \tau_N \cap \tau) - V^{\varepsilon}_N(x^{\varepsilon}(\tau_0 \cap \tau_N), \; \tau_0 \cap \tau)$$

$$= E^{\varepsilon}_{\tau_0 \cap \tau_N} \int_{\tau_0 \cap \tau_N}^{\tau_1 \cap \tau_N \cap \tau} \hat{A}^{\varepsilon} V^{\varepsilon}_N(x^{\varepsilon}(s), s) ds$$

$$\leq - \frac{\alpha_0}{2} E^{\varepsilon}_{\tau_0 \cap \tau_N} [(\tau_1 \cap \tau_N \cap \tau) - (\tau_0 \cap \tau_N)] \quad .$$

The N subscript on V^{ε}_N can be dropped, since the x- argument is always in S_N. Now, let $N \to \infty$ (hence $\tau_N \to \infty$, since by (B2), there is no finite escape time for any $\varepsilon > 0$). Then let $\tau \to \infty$. Then (4.13) and the fact that $V^{\varepsilon}(x, t) \geq - K\varepsilon$ yield (for small ε)

$$(4.14) \quad E^{\varepsilon}_{\tau_0}(\tau_1 - \tau_0) \leq 2[V^{\varepsilon}(x^{\varepsilon}(\tau_0), \tau_0) + K\varepsilon]/\alpha_0$$

$$\leq 3[V(x^{\varepsilon}(\tau_0)) + K\varepsilon]/\alpha_0 \; ,$$

and

$$- K\varepsilon + \lambda P^{\varepsilon}_{\tau_0} \{ \sup_{\tau_1 \geq s \geq \tau_0} V^{\varepsilon}(x^{\varepsilon}(s), s) \geq \lambda \} \leq |V^{\varepsilon}(x^{\varepsilon}(\tau_0), \tau_0)| \; ,$$

from which we get

$$(4.5) \quad P_{\tau_0} \left\{ \sup_{\tau_1 \geq s \geq \tau_0} V(x^{\varepsilon}(s)) \geq \lambda \right\} \leq \frac{2V(x^{\varepsilon}(\tau_0)) + K\varepsilon}{\lambda - K\varepsilon} \quad .$$

Inequality (4.14) implies that the mean travel time from ∂Q_1 to ∂Q_0 for $x^{\varepsilon}(\cdot)$ is bounded by $3(\lambda_1 + K\varepsilon)/\alpha_0$. The proof is

completed as in Theorem 2 with (4.14) and (4.15) replacing (3.3)
and (3.4), respectively.

<div align="right">Q.E.D.</div>

<u>Remark on Theorem</u> 3 <u>for non-smooth</u> F <u>and</u> G. Let $\xi(\cdot)$ be
Markov with transition function $P(\xi, t, \Gamma)$ and invariant measure
$P(\Gamma)$. Then even when $F(\cdot, \xi)$ and $G(\cdot, \xi)$ are not suitably dif-
ferentiable, the constructed $V_N^\varepsilon(\cdot)$ might still be in $\mathscr{D}(\hat{A}^\varepsilon)$
and the bounds (4.6) and (4.8) might hold. For this non-smooth
case, we write the derivative which appears on the right side of
(4.7) as

$$(4.16) \qquad \left[\int V_{N,x}'(x)\ G(x, \xi_1)\ P(\xi^\varepsilon(t),\ \frac{s}{\varepsilon^2},\ d\xi_1) \right]_x .$$

This gradient might exist for $s > 0$, even though $G(\cdot, \xi)$ is not
smooth. Similarly for the other terms. In particular write the
integrand in (4.10) in the form

$$(4.17) \qquad \int \left[\int V_{N,x}'(x)\ F(x, \xi_2)\ P(\xi_1, \frac{s}{\varepsilon^2}, d\xi_2) \right]_x'\ F(x, \xi_1)\ P(\xi^\varepsilon(t), \frac{\tau}{\varepsilon^2}, d\xi_1)$$

$$- \int \left[\int V_{N,x}'(x)\ F(x, \xi_2)\ P(\xi_1, \frac{s}{\varepsilon^2}, d\xi_2) \right]_x'\ F(x, \xi_1)\ P(d\xi_1) .$$

In any case, if with these representatives, the constructed $V_N^\varepsilon(\cdot)$
is in $\mathscr{D}(\hat{A}^\varepsilon)$ and (4.6) and (4.8) hold, then Theorem 3 continues to
hold, if $F(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are measurable but do not satisfy the
continuity and differentiability assumptions in (B2).

5. Unbounded Noise

Suppose that (1.1) is used and that the $\xi(\cdot)$ there is Markov, but <u>unbounded</u>. Then Theorem 3 is not directly valid. But, frequently in applications cases arise which can be treated by essentially the same proof, due to the special way in which the noise enters, and the properties of the dynamical terms. We now describe one such case - which was abstracted from the situation arising in several examples when one tries to apply the method of Theorem 3. Many important examples fit the situation to be described. See, for example, Example 2 of Section 6, which is typical of a large class of such cases.

We suppose that the $V_N^\epsilon(\cdot)$ constructed in Theorem 3 are in $\mathscr{D}(\hat{A}^\epsilon)$ for $\epsilon \leq \epsilon_0$, where $\epsilon_0 > 0$, and that (1.3) holds and that there are random variables $\tilde{\xi}^\epsilon(t)$, $\bar{\xi}^\epsilon(t)$, integers p,q and functions $\tilde{V}(\cdot)$ and $\bar{V}(\cdot)$ satisfying (5.1). For the first two lines of (5.1), let $x \epsilon S_N$. Let $v^\epsilon(t,\cdot)$ denote the measure of $(x^\epsilon(t),\xi^\epsilon(t))$ and let E_0 denote expectation conditioned on $(\xi^\epsilon(0),x^\epsilon(0))$.

$$\hat{A}^\epsilon V_N^\epsilon(x,t) + \mathscr{L}V(x) + 0(\epsilon)(1 + V(x)) + 0(\epsilon)\,\tilde{\xi}^\epsilon(t)\,\tilde{V}(x,t)$$

$$V_N^\epsilon(x,t) + V_N(x) + 0(\epsilon)(1 + V(x)) + 0(\epsilon)\,\bar{\xi}^\epsilon(t)\,\bar{V}(x,t)$$

(5.1)

$$\sup_{t,\epsilon} E|\tilde{\xi}^\epsilon(t)|^p < \infty \ , \ \sup_{t,\epsilon} E|\bar{\xi}^\epsilon(t)|^q < \infty$$

$$|\tilde{V}(x,t)|^{p/p-1} = 0(V(x)) = |\bar{V}(x,t)|^{q/q-1} \quad \text{for large } |x| \ .$$

Example 2 in Section 6 describes an important class where these conditions hold. For another example, consider the case where $\xi(\cdot)$ is Gauss-Markov and $F(x,\xi) = F_0(x)\xi$, $G(x,\xi) = G_0(x)\xi$ and $F_0(x)$, $G_0(x)$ as bounded and smooth and $V_x'(x)\, \bar{G}(x) \leq -\gamma V(x) + K$, $\gamma > 0$.

We shall need

C1. $\bar{G}(\cdot)$, $F(\cdot,\cdot)$, $G(\cdot,\cdot)$ <u>are measurable and are</u> $O(|x|)$ <u>for large</u> $|x|$, <u>uniformly</u> <u>bounded</u> ξ-sets.

C2. <u>For the given sequence</u> $\{x^\varepsilon(0)\}$, $\sup\limits_{\varepsilon \leq \varepsilon_0} EV(x^\varepsilon(0)) < \infty$.

C3. $\sup\limits_{t} E|\xi(t)| < \infty$. <u>For small</u> ε, $\{x^\varepsilon(\cdot), \xi^\varepsilon(\cdot)\}$ <u>and</u> $\xi(\cdot)$ <u>are Markov-Feller processes with right continuous paths and homogeneous transition functions.</u>

C4. <u>There are</u> $\gamma > 0$ <u>and</u> $K_1 < \infty$ <u>such that</u> $\mathscr{L}V(x) \leq -\gamma V(x) + K_1$.

<u>Remark.</u> Condition (C2) facilitates the proof but is not necessary for the result.

<u>Theorem 4.</u> <u>Assume</u> (C1)-(C4), (A4), (B3), (5.1) <u>and the conditions above</u> (5.1). <u>There is an</u> $\varepsilon_1 > 0$ <u>such that for</u> $\varepsilon \leq \varepsilon_1$, $(x^\varepsilon(\cdot)$, $\xi^\varepsilon(\cdot))$ <u>has an invariant measure</u> $v^\varepsilon(\cdot)$. <u>The x-marginals</u> $\{\mu^\varepsilon(\cdot)\}$ <u>of any such sequence of invariant measure converge weakly to</u> $\mu(\cdot)$ <u>as</u> $\varepsilon \to 0$. <u>Also</u> (A5) <u>and the conclusions of Theorem 1 hold.</u>

<u>Remark</u>. If $F(\cdot,\cdot)$ or $G(\cdot,\cdot)$ are not smooth, see the remark for Theorem 3. The theorem is an extension of the result in [2, Section 6] in that both use (C4).

<u>Proof</u>. Define τ_0 and τ_N as in Theorem 3. By the hypotheses, for any positive T,

$$(5.2) \quad EV^\varepsilon(x^\varepsilon(T\cap\tau_N),\ T\cap\tau_N) - EV^\varepsilon(x^\varepsilon(0),0)$$

$$= E \int_0^{T\cap\tau_N} \hat{A}^\varepsilon V^\varepsilon(x^\varepsilon(s),s)ds$$

$$\leq E \int_0^{T\cap\tau_N} [\mathscr{L}V(x^\varepsilon(s)) + O(\varepsilon) + O(\varepsilon)V(x^\varepsilon(s))$$

$$+ O(\varepsilon)|\tilde{\xi}^\varepsilon(s)\tilde{V}(x^\varepsilon(s),s)|]ds \quad .$$

By (C1), as $N \to \infty$ we have $\tau_N \to \infty$ w.p.1. The limit as $N \to \infty$ of the right side of (5.2) is bounded above by

$$(5.3) \quad - \gamma\int_0^T EV(x^\varepsilon(s))(1 + O(\varepsilon))ds$$

$$+ O(\varepsilon) \int_0^T E^{p-1/p} |\tilde{V}(x^\varepsilon(s),s)|^{p/p-1} E^{1/p}|\tilde{\xi}^\varepsilon(s)|^p\ ds + K_2T$$

$$\leq - \gamma(1 - O(\varepsilon)) \int_0^T EV(x^\varepsilon(s))ds + 2K_2T$$

for some real K_2.

Similarly the limit as $N \to \infty$ of the right side of (5.2) is bounded below by

(5.4) $EV(x^\varepsilon(T))(1 - O(\varepsilon)) - K_3$

for some real K_3. Thus for some $\gamma_1 > 0$,

(5.5) $EV(x^\varepsilon(T)) \leq K_3 - \gamma_1 \int_0^T EV(x^\varepsilon(s))ds$

$+ 2K_2 T.$

By (5.5), $EV(x^\varepsilon(T))$, $T \geq 0$, $\varepsilon \leq$ some small $\varepsilon_1\}$ is bounded by

(5.6) $EV(x^\varepsilon(T)) \leq K_3 e^{-\lambda_1 T} + 2K_2/\lambda_1$.

This and (C3) and a Theorem of Benes [11] imply that for small ε there is an invariant measure $v^\varepsilon(\cdot)$ for $(x^\varepsilon(\cdot), \xi^\varepsilon(\cdot))$ and the first assertion of the Theorem is proved. By (5.6), the x-marginals $\{\mu^\varepsilon(\cdot)\}$ of $\{v^\varepsilon(\cdot)\}$ are tight. By this and (C3) so is $\{v^\varepsilon(\cdot)\}$. Also, by (5.6) and the properties of $V(\cdot)$, $\{x^\varepsilon(t), \varepsilon \leq \varepsilon_1, t \geq 0\}$ is tight. Hence (A5) and the conclusions of Theorem 1 hold.

Now, we prove the middle assertion of the Theorem. Let $v^\varepsilon(\cdot)$ be the measure of $(x^\varepsilon(0), \xi^\varepsilon(0))$, and let $x(\cdot)$ denote the weak limit of any weakly convergent subsequence $\{x^\varepsilon(\cdot)\}$. By the invariance of $v^\varepsilon(\cdot)$, the distribution of $x^\varepsilon(t)$ is $\mu^\varepsilon(t)$ for each t. Thus, $x(t)$ has the same distribution for each t.

This must be $\mu(\cdot)$, by the uniqueness of the invariant measure $\mu(\cdot)$. In fact, if $\{v_1^\varepsilon(\cdot)\}$ denotes any sequence of invariant measures for $\{x^\varepsilon(\cdot), \xi^\varepsilon(\cdot)\}$ for which the x-marginals $\{\mu_1^\varepsilon(\cdot)\}$ are tight, then we get the same result.

We complete the proof by showing that any sequence $\{v_1^\varepsilon(\cdot)$, small $\varepsilon\}$ of invariant measures must be tight. Suppose that $\{v_1^\varepsilon(\cdot)\}$ is not tight. Then for some sequence $\varepsilon \to 0$ there are $N_\varepsilon \to \infty$ and a $\delta > 0$ and (for each ε in the sequence) a sequence $t_n^\varepsilon \to \infty$ as $n \to \infty$ and an x_ε for which

$$(5.7) \qquad P\{|x^\varepsilon(t_n^\varepsilon)| \geq N_\varepsilon | x^\varepsilon(0) = x_\varepsilon\} \geq \delta$$

(The measures of $\xi^\varepsilon(t)$ are all the same, since $\xi^\varepsilon(0) = \xi(0)$, and the ξ-marginals of $v_1^\varepsilon(\cdot)$ are all the same.) But (5.6) implies that $\varlimsup_{t\to\infty} E[V(x^\varepsilon(T))|x^\varepsilon(0) = x_\varepsilon] \leq 2K_2/\lambda_1$, a bound which depends neither on x_ε or on ε. This contradicts the assertions that $N_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $t_n^\varepsilon \to \infty$ as $n \to \infty$. Thus $\{v_1^\varepsilon(\cdot)\}$ must be tight.

Q.E.D.

6. Examples

The two examples below arise in applications to control and communication theory. They were chosen to illustrate the general results and applicability of the methods of Theorems 1,2,4 under slightly different conditions.

1. Adaptive antenna arrays. Let $v(\cdot)$ denote a complex R^r valued, stationary, bounded and right continuous stochastic process and define $M(t) = v^*(t) v'(t)$, $M^\epsilon(t) = M(t/\epsilon^2)$, $\bar{M} = EM(t)$, where * denotes the complex conjugate. Let S be a complex R^r -valued quantity and G_0 , G, τ positive real numbers. Let $\bar{W}(\cdot)$ and $W(\cdot)$ denote the solutions to

$$\tau\dot{\bar{W}} + (G\bar{M} + I)\bar{W} = G_0 S^* ,$$

$$\tau\dot{W}^\epsilon + (GM^\epsilon + I)W^\epsilon = G_0 S^* , \quad \bar{W}(0) = W^\epsilon(0) .$$

Such equations arise in the study of adaptive antenna arrays [14]. The function $\bar{W}(t)$ converges as $t \to \infty$. Define $x^\epsilon(\cdot) = (W^\epsilon(\cdot) - \bar{W}(\cdot))/\epsilon$. Then

$$(6.1) \quad \dot{x}^\epsilon = -\frac{1}{\tau}(G\bar{M} + I)x^\epsilon - \frac{G}{\tau}\delta M^\epsilon x^\epsilon - \frac{G}{\tau}\frac{\delta M^\epsilon}{\epsilon}\bar{W} , \quad x^\epsilon(0) = 0 ,$$

where $\delta M^\epsilon = M^\epsilon - \bar{M}$. The asymptotic properties (large t, small ϵ) are of interest.

Define the operator A_t^o by

$$A_t^o f(x) = \frac{G^2}{\tau^2} \int_0^\infty \bar{W}'(t) E\delta M'(s) f_{xx}(x)\delta M(0)\bar{W}(t)ds ,$$

Let $\delta M(\cdot)$ satisfy (E_t denotes expectation conditioned on $v(\rho), \rho \le t$)

$$(6.2) \quad \int_0^\infty |E_t \delta M(t + s)| \, ds \leq K, \quad \text{all } \omega, t \, ,$$

$$\int_0^\infty ds \int_0^\infty dr |E_t \delta M(t + s) \delta M'(t + s + r) - E\delta M(t + s)\delta M'(t + s + r)| \leq K \, ,$$

$$\text{all } \omega, t \quad .$$

Then $x^\epsilon(\cdot)$ is tight and converges [12] weakly to the nonhomogeneous diffusion $x(\cdot)$ with generator $\mathscr{L}$ given by

$$(6.3) \quad (\frac{\partial}{\partial t} + \mathscr{L})f(x,t) = \frac{\partial f(x,t)}{\partial t} - f_x'(x,t)(G\bar{M} + I)/\tau + A_t^{\circ} f(x,t)$$

As $t \to \infty$, the sequence of measures $\mu(t,\cdot)$ of $x(t)$ converge weakly to the invariant measure of the stationary diffusion which is obtained when $\bar{W}(\infty)$ replaces $\bar{W}(t)$ in the definition of A_t°. Define $V(x) = x'x$. Then Theorem 3 holds with the condition (6.2) replacing the mixing condition. The proof goes through owing to the fact that $\delta M^\epsilon(\cdot)$ appears linearly in (6.1).

Example 2. Weak convergence of (6.4) was investigated in [13].

$$(6.4) \quad \dot{x}^\epsilon = H(x^\epsilon) + Dy^\epsilon$$

$$y^\epsilon = Lg_\alpha(s + n^\epsilon - J(x^\epsilon))/\epsilon \, , \quad x^\epsilon(0) = x_0 \quad .$$

In (6.4), $g_\alpha(\cdot)$ is an approximation to a "hard limiter;" a function $g(\cdot)$ is a hard limiter if $g(u) = \text{sign } u, u \neq 0$. Such functions occur frequently in applications in control and communications. Instead of working with $g(\cdot)$ directly, we worked in

[13] with the approximation $g_\alpha(\cdot)$, where $g_\alpha(0) = 0$, sign $g_\alpha(u)$ = sign u, $g_\alpha(u)$ = sign u for $|u| \geq \alpha$, $|g_\alpha(u)| \leq 1$ and $\dot{g}_\alpha(u) \leq K/\alpha$ for $|u| \leq \alpha$. The process $s(\cdot)$ is a bounded right continuous signal which is a Feller-Markov (and ergodic) process with a homogeneous transition function. Let $n^\varepsilon(t) = z(t/\varepsilon^2)/\varepsilon$, where $z(\cdot)$ is a scalar valued Gaussian process with correlation function $\sigma^2 e^{-at}$, a > 0. With $H(\cdot)$ continuous and $J(\cdot)$ twice continuously differentiable, $\{x^\varepsilon(\cdot)\}$ is tight and converges weakly (as $\varepsilon \to 0$, $\alpha \to 0$ such that $\varepsilon^{\frac{1}{4}}/\alpha^{\frac{1}{2}} \to 0$) to the $x(\cdot)$ of (6.5), provided that (6.5) has a unique solution on $[0,\infty)$ (in the weak sense) for each initial condition. Now, assume this uniqueness. Assume, in addition, that $J(\cdot)$ and its partial derivatives up to order 2 are bounded and continuous, and that $H(x) = O(|x|)$ for large $|x|$. Let $V(x) = x'Qx$ for $Q > 0$ and suppose that $V'_x(x) H(x) \leq - \gamma V(x) + K$ for some $\gamma > 0$.

$$dx = H(x)dt + LD \frac{(s - J(x))}{\sigma} \sqrt{\frac{2}{\pi}} dt + LD \sqrt{\frac{2\ln 2}{a}} dB ,$$
(6.5)

$$x(0) = x_0 , \quad B(\cdot) = \text{standard Brownian motion.}$$

The perturbed Liapunov function $V_N^\varepsilon(\cdot)$ required for this case can be constructed in a very similar way to that given in Theorem 3. See, e.g., the way the analogous $f^{\varepsilon,N}$ is constructed in [13]. We then get the situation of (5.1), where $\tilde{V}(x)$ and $\bar{V}(x)$ are $O(|x|)$, and $\xi^\varepsilon(t)$, $\tilde{\xi}^\varepsilon(t)$ can each be taken to be of the form $|z(t/\varepsilon^2)|^3$. The $O(\varepsilon)$ in (5.1) need to be replaced by $O(\varepsilon^{\frac{1}{4}}/\alpha^{\frac{1}{4}})$. But if we let $\alpha \to 0$ as $\varepsilon \to 0$ such that $\varepsilon^{\frac{1}{4}}/\alpha^{\frac{1}{2}} \to 0$, Theorem 4 remains valid if $x(\cdot)$ satisfies the conditions of that theorem.

This example is a simple form of a large class that occurs naturally in control and communication theory, where the noise is unbounded, the dynamics non-linear and to which our method can be applied. The general ideas of Theorems 3 and 4 remain valid, but the actual details of proof might depend on the special conditions of the application.

7. Discrete Parameter State Dependent Noise Case.

The method of Theorems 1, 3 and 4 is readily extended to the discrete parameter case, and we will outline the idea for a case where the noise depends on the state. See [9], [10] for examples of such a case. Define $\{X_n^\epsilon\}$ by

$$(7.1) \qquad X_{n+1}^\epsilon = X_n^\epsilon + \epsilon \bar{G}(X_n^\epsilon) + \epsilon G(X_n^\epsilon, \xi_n^\epsilon) + \sqrt{\epsilon}\, F(X_n^\epsilon, \xi_n^\epsilon) + o(\epsilon) \ ,$$

where $\{\xi_n^\epsilon\}$ is a bounded noise process. In many important cases, $\{\xi_n^\epsilon\}$ depends on $\{X_n^\epsilon\}$ ([10]). We model this in the following way. Let $\{X_n^\epsilon, \xi_n^\epsilon\}$ be a homogeneous Markov process with the one step marginal transition function written as

$$P^\epsilon(\xi, 1, \Gamma | x) = P\{\xi_1^\epsilon \in \Gamma | X_0^\epsilon = x, \xi_0^\epsilon = \xi\} \ \ .$$

For each x, define the bounded chain $\{\xi^\epsilon(x)\}$ by the transition function defined recursively by

$$P^\epsilon(\xi, j, \Gamma | x) = \int P^\epsilon(\xi, j - \ell, d\xi_1 | x)\, P^\epsilon(\xi_1, \ell, \Gamma | x) \ \ .$$

Let $\{\xi^\epsilon(x)\}$ have a unique invariant measure $\bar{P}^\epsilon(\cdot)$ and let $\bar{E}^\epsilon$ denote expectation of functionals of the chain under the stationary measure.

The comments below are formal, and it is implicitly assumed that the indicated derivatives and sums exist. Suppose that there are continuous $a_{ij}(\cdot)$ and $\bar{F}(\cdot)$ such that for each x the limits (7.2) and (7.3) exist for smooth $f(\cdot)$. (The idea is simpler for the case of non-state dependent noise, but owing to the

numerous applications, it is worthwhile to present the more general case.)

$$(7.2) \quad \sum_{\ell=1}^{\infty} \bar{E}^{\epsilon}[E(F(x,\xi_{\ell}^{\epsilon}(x))|\xi_0^{\epsilon}(x))]_x \ F(x,\xi_0^{\epsilon}(x)) \rightarrow \bar{F}(x)$$

$$\frac{1}{2} \bar{E}^{\epsilon}F'(x,\xi_0^{\epsilon}(x)) \ f_{xx}(x) \ F(x,\xi_0^{\epsilon}(x))$$

$$(7.3) \quad + \sum_{\ell=1}^{\infty} \bar{E}^{\epsilon}F'(x,\xi_{\ell}^{\epsilon}(x)) \ f_{xx}(x) \ F(x,\varsigma_0^{\epsilon}(x)) \rightarrow$$

$$\frac{1}{2} \sum_{i,j} a_{ij}(x) \ f_{x_i x_j}(x) \quad .$$

Define $\mathscr{L}$ by

$$(7.4) \quad \mathscr{L} = \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i [\bar{F}_i(x) + \bar{G}_i(x)] \frac{\partial}{\partial x_i} \quad .$$

Let $x^{\epsilon}(\cdot)$ denote the piecewise constant process with values X_n^{ϵ} on $[\epsilon n, \epsilon n + \epsilon)$. Under some additional conditions, $\{x^{\epsilon}(\cdot)\}$ converges weakly to the diffusion $x(\cdot)$ with generator $\mathscr{L}$ [9,10].

For the discrete case, the operator $\hat{A}^{\epsilon}$ is defined on the set of functions $\mathscr{F}$ which are constant on $[n\epsilon, n\epsilon + \epsilon)$, and $\{X_j^{\epsilon}, j \leq n, \xi_j^{\epsilon}, j < n\} \equiv \mathscr{S}_n^{\epsilon}$ measurable at $n\epsilon$. For $f(\cdot) \in \mathscr{F}$, $\hat{A}^{\epsilon}f$ is defined by $\hat{A}^{\epsilon}f(t) = [E_t^{\epsilon}f(t+\epsilon) - f(t)]/\epsilon$.

Given a Liapunov function $V(\cdot)$, the perturbed and truncated $V_N^{\epsilon}(\cdot)$ are found by a method that is very similar to that used in Theorem 3, but with summation replacing integration, and a truncated Taylor series expansion rather than a differentiation used to get the $\hat{A}^{\epsilon}V_{N,i}^{\epsilon}(\cdot)$. Perturbed test functions are constructed in

[9], [10] and are used in the proof of the weak convergence mentioned above (following the method of [5]). The $V_{N,i}^{\varepsilon}(\cdot)$ would be constructed just as these perturbed test functions were, but using test function $V_N(\cdot)$. If the $V_N^{\varepsilon}(\cdot) - V(\cdot)$ and $\hat{A}^{\varepsilon}V_N^{\varepsilon}(\cdot)$ $- \mathcal{L}V(\cdot)$ satisfy the bounds (4.6), (4.8) in the set S_N, then the discrete parameter analog of Theorem 3 holds. Theorem 1 holds in any case if the $\{x^{\varepsilon}(\cdot)\}$ of this section satisfies the conditions of that theorem.

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